DARBOUX TRANSFORMATIONS

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1. **Introduction.** A real valued function of a real variable is said to be a Darboux function if it maps every connected set in its domain onto a connected set. Darboux functions have been studied by many authors. A detailed account of such functions, along with an extensive bibliography, can be found in the survey article [1]. An important fact about the class of Darboux functions is that it contains many important classes of functions, for example, the class of approximately continuous functions as well as the classes of derivatives and approximate derivatives.

The notion of Darboux function has been generalized to transformations whose domain and/or range are topological spaces more general than the real line. A generalized Darboux transformation is then one which transforms each set of a suitably chosen family $\mathscr{A}$ of connected sets onto a connected set [4], [5], [8], [10], [11], [12]. Various families $\mathscr{A}$ have proved natural for different purposes. If $\mathscr{A}$ is taken to be the family of *all* connected sets, one arrives at the notion of connected function [4], [12]. This family, however, is too large if one desires the approximately continuous transformations or certain types of (generalized) derivatives to be included within the class of Darboux transformations [5], [8], [11], or if one wishes certain theorems on Darboux functions to extend to the more general setting. For these purposes, certain smaller families $\mathscr{A}$ are appropriate [5], [10], [11]. These families consist of sets which are closures of elements of certain bases $\mathscr{B}$ of the domain space.

In this article we study a notion of Darboux transformation which includes, as special cases, the notions considered in [5] and [11]. Our notion is similar to the one considered in [10], except that we do not impose the blanket restriction found in [10] that the transformations be real valued. In §§3 and 4 we consider statements analogous to various theorems found in [2], [3], [9], [10], [13], and [14]. We show that these analogous statements are correct in our general setting provided certain mild restrictions are imposed on the base $\mathscr{B}$. We also give examples to show that these restrictions cannot be dropped, where they appear, without falsifying the statements. Finally, in §5 we show that the usual conditions which imply that a Darboux function be continuous do not carry over to our setting, and we give a necessary and sufficient condition that a Darboux transformation be continuous.

2. **Preliminaries.** Throughout this article X will be a euclidean space, X^* a

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separable, metric space and $\mathscr{B}$ a (topological) base for X such that every $U \in \mathscr{B}$ is connected. A transformation f mapping X into X^* is said to be a *Darboux transformation* relative to the base $\mathscr{B}$, or, briefly, Darboux ($\mathscr{B}$) provided $f(\overline{U})$ is connected in X^* for every $U \in \mathscr{B}$. We note that if X and X^* are the real line and $\mathscr{B}$ is the base of all open intervals, then our notion of a Darboux ($\mathscr{B}$) transformation reduces to the ordinary notion of a Darboux function.

It is clear that if f is a Darboux transformation relative to a base $\mathscr{B}$, then f maps every open, connected set onto a connected set. More generally, if E is any set which can be represented as a union of closures of base elements $E = \bigcup \overline{U}_\alpha$, $U_\alpha \in \mathscr{B}$, such that for $x \in E$ and $y \in E$ there exists a finite subcollection of these base elements, $U_1, \ldots, U_n$ with $x \in \overline{U}_1$, $y \in \overline{U}_n$, and for $k = 1, 2, \ldots, n-1$, $\overline{U}_k \cap \overline{U}_{k+1} \neq \varnothing$, then $f(E)$ is connected. It is easy to construct examples, however, of transformations which are Darboux relative to one base, but not Darboux relative to another. We shall sometimes require that $\mathscr{B}$ satisfy certain conditions.

DEFINITION. A base $\mathscr{B}$ for X is said to satisfy condition (*) provided any translation of an element of $\mathscr{B}$ is in $\mathscr{B}$. The base $\mathscr{B}$ satisfies condition (**) provided for $x \in X$ and $U \in \mathscr{B}$, $x \in \overline{U}$, there exists $V \in \mathscr{B}$ such that $x \in \overline{V}$ and $\overline{V} - \{x\} \subset U$.

3. **A local characterization.** In this section we prove a theorem which can be interpreted as a local characterization of real valued, Darboux transformations. Our theorem generalizes a result found in [2]. It would be of interest to know whether such a theorem exists in case X^* is not restricted to being the real line. We mention that several candidates for such a theorem fail.

In the statement of Theorem 1, below, the expression $\liminf_{x \to x_0(U)} f(x)$ indicates that the limit inferior is taken with x restricted to the set $\{x_0\} \cup U$. A similar interpretation applies to the expression $\limsup_{x \to x_0(U)} f(x)$.

THEOREM 1. *Let X be a euclidean space and let $\mathscr{B}$ be a base of connected sets for X which satisfies conditions* (*) *and* (**). *Let f be a real valued function defined in X. Then f is Darboux ($\mathscr{B}$) if and only if for every $x_0 \in X$ and every $U \in \mathscr{B}$ such that $x_0 \in \overline{U}$, the inclusion* $(\liminf_{x \to x_0(U)} f(x), \limsup_{x \to x_0(U)} f(x)) \subset f(\overline{U})$ *holds.*

Proof. That the condition is necessary, is obvious. To prove the sufficiency of the condition, we argue by contradiction. Suppose, then, that f satisfies the condition, but is not Darboux ($\mathscr{B}$). There is a $U \in \mathscr{B}$ such that $f(\overline{U})$ is not connected. Thus there is a number γ such that $\overline{U} \cap \{x : f(x) = \gamma\}$ is empty but each of the sets $A \equiv \overline{U} \cap \{x : f(x) > \gamma\}$ and $B \equiv \overline{U} \cap \{x : f(x) < \gamma\}$ is not empty. Let P be the boundary of A in $\overline{U}$. We show first that the set $P \cap U$ is a nonempty, dense-in-itself set of type G_δ. Towards this end, we note that P is not empty, for otherwise A and B would form a separation of the connected set $\overline{U}$. Suppose $P \cap U = \varnothing$. Then $P \subset \overline{U} - U$, and $U \subset A$ or $U \subset B$, say $U \subset A$. Let $b \in B$. Then $b \in \overline{U} - U$. By condition (**), there is a $V \in \mathscr{B}$ such that $b \in \overline{V}$ and $\overline{V} - \{b\} \subset U \subset A$. Since $f(b) < \gamma$ and $f(a) > \gamma$ for all $a \in A$, the requirement that the interval $(\liminf_{x \to b(V)} f(x)$, $\limsup_{x \to b(V)} f(x))$ be contained in $f(\overline{V})$ is violated. It follows that $P \cap U \neq \varnothing$.

Now suppose $P \cap U$ contains an isolated point p, say $p \in P \cap A$. Let V be a neighborhood of p such that $V \subset U$ and $V \cap P = \{p\}$. If X is of dimension $\geqq 2$, then $V-\{p\}$ is connected and we deduce readily that $V-\{p\} \subset A$ or $V-\{p\} \subset B$. That the first inclusion fails follows immediately from the definition of P, and that the second inclusion is impossible follows readily from the hypothesis of the theorem. Thus P is dense-in-itself if X is of dimension $\geqq 2$. A simple argument shows the same result holds if the dimension of X is 1.

Since P is closed and U is open, $P \cap U$ is of type G_δ. We have shown that $P \cap U$ is a nonempty, dense-in-itself set of type G_δ.

Let $P_1 = P \cap U$. Both of the sets $A \cap P_1$ and $B \cap P_1$ are dense in P_1. To verify this, suppose, for example, that $A \cap P_1$ fails to be dense in P_1. Let $V \subset U$ be an open sphere centered at a point $p_0 \in P_1 \cap B$ such that $V \cap A \cap P_1 = \varnothing$. Then $A \cap V$ is open and nonempty. Let x be a point of $A \cap V$ such that $\rho(x, X-V) > 2\rho(x, p_0)$, where ρ is the euclidean metric of X. Choose $W \in \mathscr{B}$ such that $x \in W \subset A \cap V$ and $\rho(\overline{W}, X-V) > \rho(\overline{W}, p_0)$. Let q be a point of B nearest x. It is clear that $q \in V$. Let W_1 be a translation of W such that $q \in \overline{W}_1 - W_1$ and $W_1 \subset A$. Since $\mathscr{B}$ satisfies condition $(*)$, $W_1 \in \mathscr{B}$. Finally, choose $W_2 \in \mathscr{B}$ such that $q \in \overline{W}_2$ and $\overline{W}_2 - \{q\} \subset W_1$. This is possible because $\mathscr{B}$ satisfies condition $(**)$. Now $q \in B$ while $\overline{W}_2 - \{q\} \subset A$, contradicting the hypothesis of the theorem. Therefore $A \cap P_1$ (and similarly $B \cap P_1$) must be dense in P_1.

We show now that if $p \in A \cap U$ and $\varepsilon > 0$, there is a neighborhood V of p such that if $x \in \overline{V}$ then $f(x) > \gamma - \varepsilon$ (with the analogous statement holding for each $p \in B \cap U$). Suppose the condition fails for some $p \in A \cap U$ and $\varepsilon > 0$. Let $V_1 \supset V_2 \supset \cdots$ be a sequence of neighborhoods of p such that $V_k \in \mathscr{B}$ for each k and $\{p\} = \bigcap_{k=1}^{\infty} V_k$. The hypotheses of the theorem guarantee that $(\gamma-\varepsilon, f(p)) \subset f(\overline{V}_k)$ for each k. In particular, for each k, there is a point $p_k \in \overline{V}_k$ such that $f(p_k) = \gamma$. But for k sufficiently large, $p_k \in U$, since $p \in U$. This contradicts the assumption that f does not assume the value γ on the set U.

We are now ready to complete the proof of Theorem 1. Let $p_1 \in A \cap P_1$. Choose $V_1 \in \mathscr{B}$ such that $p_1 \in V_1$, the diameter of $V_1 < 1$, $\overline{V}_1 \subset U$, and if $p \in \overline{V}_1$ then $f(p) > \gamma - 1$. Since $B \cap P_1$ is dense in P_1, there is a point $p_2 \in B \cap P_1 \cap V_1$. Let $V_2 \in \mathscr{B}$ such that $p_2 \in V_2$, $V_2 \subset V_1$, the diameter of $V_2 < \frac{1}{2}$, and if $p \in \overline{V}_2$, then $f(p) < \gamma + \frac{1}{2}$. We continue in this manner obtaining a sequence of points $\{p_k\}$ and a sequence of neighborhoods $\{V_k\}$ such that for all k, the following conditions are satisfied: $p_k \in V_k \cap P_1$; $V_k \in \mathscr{B}$; $V_{k+1} \subset V_k$; the diameter of V_k is less than $1/k$; and if $p \in \overline{V}_k$ and k is odd (even) then $f(p) > \gamma - 1/k$ ($f(p) < \gamma + 1/k$, resp.). The set $\bigcap_{k=1}^{\infty} \overline{V}_k$ consists of a single point p_0. Since $\overline{V}_1 \subset U$, $p_0 \in U$. It follows from the definition of the sets V_k that $f(p_0) = \gamma$. This contradicts the assumption that γ is not assumed on U.

The proof of Theorem 1 is complete.

In the proof of Theorem 1, we made use of conditions $(*)$ and $(**)$. We now give an example to show that Theorem 1 would fail if we deleted either of these conditions.

EXAMPLE. Let X be the euclidean plane. Let f be a function defined on X and satisfying the conditions that $f(x, y)=y$ if $y\neq 0$, and f maps every interval of the x-axis onto the interval $(-\infty, 0)$. Let $\mathscr{B}_1$ be the base of all open discs *not* tangent to the x-axis and let $\mathscr{B}_2$ be the base of all open squares with sides parallel to the coordinate axes. The base $\mathscr{B}_1$ does not satisfy condition $(*)$ while the base $\mathscr{B}_2$ does not satisfy condition $(**)$. It is easy to verify that f satisfies the hypotheses of Theorem 1 relative to the bases $\mathscr{B}_1$ and $\mathscr{B}_2$ but is not Darboux $(\mathscr{B}_1)$ nor Darboux $(\mathscr{B}_2)$. In particular we note that (using the notation of the proof of Theorem 1) for $\gamma=0$, and for any U in $\mathscr{B}_1$ (or $\mathscr{B}_2$) such that $\bar{U}$ intersects the x-axis, the set P is the intersection of $\bar{U}$ with the x-axis and $P\subset B$ (so that $A \cap P$ fails to be dense in P). It is possible to choose $U \in \mathscr{B}_2$ such that $P \cap U=\varnothing$, i.e., so that P is entirely contained in the boundary of U. These results should be compared with the relevant parts of the proof of Theorem 1.

4. Darboux transformations of Baire type 1.

The Darboux functions of Baire type 1 play an important role in the theory of functions of a real variable because many classes of functions are contained in this class. For example, the class of approximately continuous functions, the class of derivatives and the class of approximate derivatives are contained in the class of Darboux Baire 1 functions. It is therefore no surprise that many authors have characterized this class. Several such characterizations (with references) can be found in [1]. We show now that the analogues of several of these characterizations are valid for Darboux transformations of Baire type 1 which map a euclidean space X into a separable metric space X^* provided appropriate restrictions are imposed on the base $\mathscr{B}$. We begin with a definition.

DEFINITION. Let X be a euclidean space and $\mathscr{B}$ a base for X. A set $S\subset X$ is *dense-in-itself* $(\mathscr{B})$ (*c-dense-in-itself* $(\mathscr{B})$) provided if $x \in S$ and $U \in \mathscr{B}$, with $x \in \bar{U}$, then $S \cap U$ contains a point other than x ($S \cap U$ has cardinality c, resp.).

The proof of the implication (iv) $\to$ (v) of Theorem 2, below, requires a condition $(**+)$ on the base $\mathscr{B}$ which is slightly stronger than condition $(**)$.

DEFINITION. The base $\mathscr{B}$ satisfies condition $(**+)$ provided for each $U \in \mathscr{B}$, $x \in \bar{U}$ and $\varepsilon>0$ there is a $V \in \mathscr{B}$ such that $x \in \bar{V}$, $\bar{V}-\{x\}\subset U$ and the diameter of V is less than ε.

Each of the conditions (ii) through (vi) appearing in the statement of Theorem 2 is the analogue of a condition considered by a previous author for ordinary Darboux functions. We provide references within the statement of the theorem.

THEOREM 2. *Let X be a euclidean space and X^* a separable metric space. Let f be a transformation of Baire type* 1 *mapping X into X^*. Let $\mathscr{B}$ be a base of connected sets for X. Consider the following conditions*:

(i) *f is Darboux $(\mathscr{B})$.*

(ii) *$[f(\bar{U})]^-$ is connected for all $U \in \mathscr{B}$* (*Ellis* [3]).

(iii) *$f(\bar{U})\subset[f(U)]^-$ for all $U \in \mathscr{B}$* (*Young* [13]).

(iv) *If V^* is open in X^*, $f^{-1}(V^*)$ is c-dense-in-itself* ($\mathscr{B}$) (*Zahorski* [14]).

(v) *If $x \in X$, $U \in \mathscr{B}$, $x \in \bar{U}$, there exists a perfect set P such that $x \in P \subset \bar{U}$ and $f|P$ is continuous at x* (*Maximoff* [9]).

(vi) *If V^* is open in X^*, $f^{-1}(V^*)$ is dense-in-itself* ($\mathscr{B}$) (*Zahorski* [14]).

Then the following chain of implications is valid, the $(*)$, $(**)$ *or* $(**+)$ *appearing above the symbol '$\to$' indicating that we are assuming the base satisfies conditions* $(*)$, $(**)$, *or* $(**+)$ *respectively in that implication*:

$$\text{(i)} \longrightarrow \text{(ii)} \xrightarrow{(**)} \text{(iii)} \xrightarrow{(*)} \text{(iv)} \xrightarrow{(**+)} \text{(v)} \longrightarrow \text{(vi)} \xrightarrow{(*)} \text{(i)}.$$

In particular, if $\mathscr{B}$ satisfies conditions $(*)$ *and* $(**+)$ *all six conditions are equivalent.*

Proof. (i) $\to$ (ii). This follows immediately from the definition of Darboux transformation.

(ii) $\to^{(**)}$ (iii). Suppose f does not satisfy (iii). There exists $U \in \mathscr{B}$ and $x \in \bar{U}$ such that $f(x) \notin [f(U)]^-$. Let V^* be a neighborhood of $f(x)$ such that $V^* \cap [f(U)]^- = \varnothing$. Condition $(**)$ implies the existence of a $W \in \mathscr{B}$ such that $\bar{W}-\{x\} \subset U$ and $x \in \bar{W}$. Now $[f(\bar{W}-\{x\})]^- \subset [f(U)]^-$ so $[f(\bar{W}-\{x\})]^- \cap V^* = \varnothing$. On the other hand, $f(x) \in V^*$. It follows that $[f(\bar{W})]^-$ is not connected, so that condition (ii) is not satisfied.

(iii) $\to^{(*)}$ (iv). Let V^* be open in X^*, let $x_0 \in f^{-1}(V^*)$ and let $U \in \mathscr{B}$ be such that $x_0 \in \bar{U}$. Since $f(\bar{U}) \subset [f(U)]^-$, there is a point x_1 in U such that $f(x_1) \in V^*$. Let V_1^* be a neighborhood of $f(x_1)$ such that $\bar{V}_1^* \subset V^*$, and choose $W \in \mathscr{B}$ such that $x_1 \in W$ and $\bar{W} \subset U$. It follows readily from condition (iii) and condition $(*)$ that the set $D \equiv W \cap f^{-1}(V_1^*)$ is dense-in-itself.

Therefore the set $\bar{D}$ is perfect. Since f is a transformation of Baire type 1 and X^* is separable, $f|\bar{D}$, the restriction of f to $\bar{D}$, is continuous on a residual subset of $\bar{D}$ which, of course, has cardinality c [7]. If x is such a point of relative continuity, then $f(x) \in \bar{V}_1^*$, because D is dense in $\bar{D}$ and $f(D) \subset V_1^*$. Now $\bar{D} \subset \bar{W} \subset U$ and $\bar{V}_1^* \subset V^*$, from which it follows that $f^{-1}(V^*) \cap U$ has cardinality c. (Thus condition (iv) is satisfied.)

(iv) $\to^{(**+)}$ (v). Let $x_0 \in X$, $U \in \mathscr{B}$, $x_0 \in \bar{U}$ and suppose (iv) is satisfied. Let V_0^* be a sphere of radius $\frac{1}{2}$ about $f(x_0)$ and W_1 be a base element in $\mathscr{B}$ such that diameter $\bar{W}_1 \leqq 1$, $\bar{W}_1-\{x_0\} \subset U$ and $x_0 \in \bar{W}_1$. By (iv), W_1 contains c points of $f^{-1}(V_0^*)$. Since f is of Baire type 1, $W_1 \cap f^{-1}(V_0^*)$ is of type F_σ and, having cardinality c, contains a perfect set P_1 [7]. Let x_1 be a point of continuity of $f|P_1$ and let U_1 be a neighborhood of x_1 such that $Q_1 \equiv \bar{U}_1 \cap P_1$ is perfect, $x_0 \notin Q_1$ and $\rho(f(x), f(x_1)) < \frac{1}{2}$ if $x \in Q_1$. We proceed by induction. Suppose we have disjoint sets $Q_1, \ldots, Q_n$ such that for $i = 1, 2, \ldots, n$, Q_i is perfect, $x_0 \notin Q_i$, $\rho(x, x_0) < 1/i$ if $x \in Q_i$, and $f(Q_i) \subset V_i^*$, where V_i^* is a sphere of radius $1/i$ about $f(x_0)$. The set $Q_1 \cup \cdots \cup Q_n$ is then a perfect subset of U which is a positive distance d from x_0. Let W_{n+1} be a base element in $\mathscr{B}$ whose diameter is less than min $(d, 1/(n+1))$ and such that $\overline{W}_{n+1}-\{x_0\}\subset U$ and $x_0\in \overline{W}_{n+1}$. The existence of W_{n+1} is guaranteed by condition $(**+)$. The set $W_{n+1}\cap f^{-1}(V^*_{n+2})$ has cardinality c. As before, it contains a perfect subset Q_{n+1} such that $x_0\notin Q_{n+1}$, $f(Q_{n+1})\subset V^*_{n+1}$ and Q_{n+1} lies in the sphere of radius $1/(n+1)$ centered at x_0.

Now let $Q=\bigcup_{n=1}^{\infty} Q_n\cup\{x_0\}$. It is easily verified that Q is perfect and that $f|Q$ is continuous at x_0. Thus, condition (v) is satisfied.

(v) $\to$ (vi). This is obvious.

(vi) $\to^{(*)}$ (i). The proof of this part is similar to the proof of Theorem 1, so we outline the proof, omitting details. Suppose f satisfies condition (vi) but is not Darboux ($\mathscr{B}$). There exists $U\in\mathscr{B}$ such that $f(\overline{U})$ is not connected. Let $f(\overline{U})=A^*\cup B^*$, where A^* and B^* are disjoint, nonempty, relatively open subsets of $f(\overline{U})$. Let $A=\overline{U}\cap f^{-1}(A^*)$, $B=\overline{U}\cap f^{-1}(B^*)$ and let P be the boundary of A in $\overline{U}$. As in the proof of Theorem 1, we show $P_1=P\cap U$ is a nonempty dense-in-itself set of type G_δ. (We do not need condition $(**)$ in this proof, however!) Using condition $(*)$ (again, condition $(**)$ is not required) we can show that the sets $A\cap P_1$ and $B\cap P_1$ are dense in P_1. But P_1 is of type G_δ and f is of Baire type 1, from which it follows that $f|P_1$ has a point of continuity [7, p. 326]. In view of the definitions of A and B, and the fact that $A\cap P_1$ and $B\cap P_1$ are dense in P_1, this is impossible. A contradiction has been established. Thus f is Darboux ($\mathscr{B}$).

The proof of Theorem 2 is now complete.

REMARK 1. Condition $(**)$ cannot be dropped from the implication (ii) $\to^{(**)}$ (iii) (see the example of §5) nor can condition $(*)$ be dropped from the implication (vi) $\to^{(*)}$ (i). To see this, let $\mathscr{B}$ be the base of open discs not tangent to the x-axis of the euclidean plane and let f be the characteristic function of the closed upper half plane. To verify that $(*)$ cannot be dropped from (iii) $\to^{(*)}$ (iv) we consider the base $\mathscr{B}$ of all open discs not having the origin as a boundary point and then let f be the characteristic function of the origin. Finally we give an example to show that $(**+)$ cannot be dropped from (iv) $\to^{(**+)}$ (v) even if $\mathscr{B}$ satisfies $(*)$ and $(**)$.

Let X be the euclidean plane furnished with the usual coordinate system. Let T_1 be the open, triangular region with vertices $(0,0)$, $(-2,4)$, and $(2,4)$ and T_2 the open, triangular region with vertices $(0,0)$, $(-1,3)$, and $(1,3)$. Let f be a real valued function such that f vanishes on T_2, has value 1 on the complement of T_1 and is continuous at every point except the origin. Let $\mathscr{B}$ be the base consistency of all open discs and the family of all triangular regions S_n with vertices $(0,0)$, $(-1/n, 5+1/n)$ and $(1/n, 5+1/n)$ for $n=1,2,\ldots$ and all translations of this family of triangles.

It is easy to verify that this base satisfies conditions $(*)$ and $(**)$ but not condition $(**+)$, and that f satisfies hypothesis (iv) but not hypothesis (v) (consider for example the base element S_1 with x_0 the origin). The difficulty, of course, lies in the artificial manner in which this base was concocted. The base of open discs causes no trouble with either hypothesis (iv) or (v) but offers no help in overcoming the difficulties caused by the collection of triangular regions in $\mathscr{B}$.

REMARK 2. Condition (v) provides a continuity theorem for Baire 1 transformations which are Darboux relative to a base satisfying conditions $(*)$ and $(**+)$. To arrive at this continuity condition, we first note that a transformation f of Baire type 1 has the property of Baire on every perfect set [7, pp. 306, 307]; that is, if Q is a perfect set in the domain of f, then there is a set R, residual in Q, such that $f|R$ is continuous. Such a set R contains a perfect set S, and $f|S$ is continuous. Consider now the sets Q_i, $i = 1, 2, \ldots$ which appeared in the proof of (iv) $\rightarrow^{(**+)}$ (v) above. For each i, let S_i be a perfect subset of Q_i such that $f|S_i$ is continuous. The set $P \equiv \{x_0\} \cup (\bigcup_{i=1}^{\infty} S_i)$ is then a perfect set containing x_0 such that $f|P$ is continuous. We state this result as a corollary.

COROLLARY. *If $\mathscr{B}$ is a base for the euclidean space X and $\mathscr{B}$ satisfies conditions* $(*)$ *and* $(**+)$, *then any Darboux* $(\mathscr{B})$ *transformation of Baire type* 1 *from X to a separable metric space X^* has the property that if $x_0 \in X$, $U \in \mathscr{B}$ and $x_0 \in \overline{U}$, there exists a perfect set P such that $x_0 \in P \subset \overline{U}$ and $f|P$ is continuous.*

REMARK 3. We need not assume that X^* is separable in the statement of Theorem 2, if we assume the continuum hypothesis. For in that case, the separability of the set $f(X)$ follows from the assumption that f is a Baire function [7, p. 305]. The proof of Theorem 2, with X^* replaced by $f(X)$ goes through without further change.

REMARK 4. In their study of approximately continuous transformations, Goffman and Waterman [5] showed, in our language, that every approximately continuous transformation of a euclidean space X into a separable metric space X^*, is of Baire type 1 and is Darboux $(\mathscr{B})$ where $\mathscr{B}$ is the base of those connected sets U with the property that if $x_0 \in \overline{U}$, then x_0 is a point of positive, upper metric density of U. It is clear that $\mathscr{B}$ satisfies condition $(*)$. Now, an approximately continuous transformation f is one with the property that if V^* is open in X^*, then every point in $f^{-1}(V^*)$ is a point of density of $f^{-1}(V^*)$. Thus their result can be stated as follows: Suppose f has the property that for any open $V^* \subset X^*$, if $x_0 \in f^{-1}(V^*)$, $U \in \mathscr{B}$ and $x_0 \in \overline{U}$, then $U \cap f^{-1}(V^*)$ has positive upper density at x_0. Then $f(\overline{U})$ is connected for every $U \in \mathscr{B}$, where $\mathscr{B}$ is the base described above. Our result (vi) $\rightarrow^*$ (i), relative to the base $\mathscr{B}$, shows that it is sufficient that f have the property $f^{-1}(V^*) \cap U$ contain a *single point* different from x_0. This distinction is analogous to Zahorski's results [14] concerning approximately continuous functions and Darboux, Baire 1 functions (of a real variable).

5. **Darboux transformations and continuity.** A Darboux function of a real variable can be everywhere discontinuous and even nonmeasurable. However, it is easily verified that if f is a Darboux function and $f^{-1}(x^*)$ is closed for each $x^* \in f(X)$, then f must be continuous. The corresponding statement for Darboux transformations is not valid, even under much stronger hypotheses. In this section we give an example of a one-to-one Darboux transformation of Baire type 1 which is not continuous, and then provide a necessary and sufficient condition for a Darboux transformation to be continuous.

EXAMPLE. Let $X=X^*$ be the euclidean plane, furnished with the usual coordinate system. Let f be defined by

$$f(x, y) = \begin{cases} (x, \sin(1/x)+y) & \text{if } x \neq 0, \\ (x, y) \quad \text{if } x = 0. \end{cases}$$

It is easy to verify that f is a one-to-one transformation of Baire type 1 which maps X onto X^*, that f is discontinuous on $Y \equiv \{(x, y) : x=0\}$, and that f is Darboux relative to the base of open disks. (Actually, it can be verified that both f and f^{-1} are of Baire type 1 and are Darboux relative to the base of *all* open connected sets.)

As we mentioned in Remark 1 following the proof of Theorem 2, this example also shows that condition (ii) of Theorem 2 does not imply condition (iii) if condition (**) is dropped. We first observe that the base of all open, connected sets in the euclidean plane does not satisfy condition (**). For example, an appropriately chosen "thin band" about the set $\{(x, y) : y=-\sin(1/x), 0<x<1\}$ defines a connected, open set U such that the origin is in $\overline{U}-U$, but there is no set V as guaranteed by condition (**). Now let f be the function of the example above. The set $f(U)$ is then a "thin" band about the interval $0<x<1$ of the x-axis, $[f(U)]^- \cap Y=\{(0, 0)\}$, where Y is the y-axis. But $\overline{U}$ contains the set $Z=\{(x, y): -1<y<1\} \cap Y$, and since $f|Y$ is the identity function, $f(\overline{U})$ contains Z. Thus $f(\overline{U}) \not\subset [f(U)]^-$. Hence condition (ii) is satisfied, but condition (iii) is not.

We turn now to our continuity theorem.

THEOREM 3. *Let X be a euclidean space and X^* a locally compact, metric space. Let $\mathscr{B}$ be a base of connected sets for X and let f be a Darboux ($\mathscr{B}$) transformation of X into X^*. A necessary and sufficient condition that f be continuous is that if T^* is the boundary of any sphere whose center is not an isolated point of X^*, then $f^{-1}(T^*)$ is closed.*

Proof. The necessity is obvious.

Suppose, now, that f is Darboux ($\mathscr{B}$), but discontinuous at x_0. Let $\{U_k\}$, $k=1, 2, \ldots$, be the sequence of open spheres centered at x_0 and such that the radius of U_k is $1/k$. Since each U_k is a connected, open set and f is Darboux ($\mathscr{B}$), the sets $f(U_k)$ are connected. (Recall that a transformation which is Darboux relative to any base transforms every open, connected set onto a connected set.)

Let X^*_∞ be X^* if X^* is compact, and the one point compactification of X^* if X^* is not compact. The sets $[f(U_k)]^-$ are then compact, connected subsets of X^*_∞; thus the same is true of the set $K^*=\bigcap_{k=1}^\infty [f(U_k)]^-$ [6, p. 163]. The set K^* contains $x_0^*=f(x_0)$. Since f is discontinuous at x_0, K^* also contains another point of X^*_∞. But since K^* is connected, there is a point $x^* \in K^* \cap X^*$, $x^* \neq x_0^*$ (because the set consisting of only x_0^* and the "point at infinity," if X^* is not compact, is not connected in X^*_∞). It is clear that neither x_0^* nor x^* can be isolated in X^*. Suppose S^* is an open sphere centered at x^*, of radius less than $\rho(x_0^*, x^*)$, such that T^*, the boundary of S^*, contains a point of X^*. Let k be a positive integer; then $x^* \in [f(U_k)]^-$. Furthermore, $f(U_k)$ contains x_0^* as well as points of S^*. It follows that $f(U_k) \cap T^*$ is not empty (otherwise $f(U_k) \cap S^*$ and $f(U_k) \cap (X^* - \bar{S}^*)$ would effect a separation of $f(U_k)$). Therefore $U_k \cap f^{-1}(T^*) \neq \varnothing$. Since this is true for every $k = 1, 2, \ldots$, and $x_0 \notin f^{-1}(T^*)$, we conclude $f^{-1}(T^*)$ is not closed.

The proof of Theorem 3 is now complete.

Under suitable restrictions on X^*, the family of boundaries of spheres can be replaced by other families. For example, if X^* is an n-dimensional euclidean space, then a Darboux transformation f into X^*, having the property that $f^{-1}(T^*)$ is closed for every $(n-1)$-dimensional hyperplane T^* perpendicular to a coordinate axis, is continuous. There is no essential change in the proof. In particular, for a real valued, Darboux function f defined on a euclidean space X, one need only know that $f^{-1}(x^*)$ is closed for each $x^* \in f(X)$ in order to be able to infer that the function is continuous. This is a direct extension of the theorem, mentioned at the beginning of this section, for real valued functions of a real variable.

References

1. A. M. Bruckner and J. G. Ceder, *Darboux continuity*, Jber. Deutsch. Math.-Verein. **67** (1965), 93–117.

2. A. Császár, *Sur la propriété de Darboux*, C. R. Premier Congrès des Mathématiciens Hongrois, pp. 551–560, Akademiai Kiado, Budapest, 1952.

3. H. Ellis, *Darboux properties and applications to non-absolutely convergent integrals*, Canad. J. Math. **3** (1951), 471–485.

4. K. Fan and R. Struble, *Continuity in terms of connectedness*, Indag. Math. **16** (1954), 161–164.

5. C. Goffman and D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. **12** (1961), 116–121.

6. J. Kelley, *General topology*, Van Nostrand, New York, 1955.

7. C. Kuratowski, *Topologie*. I, 4th ed., Monogr. Mat. Vol. 20, Warsaw, 1958.

8. S. Marcus, *A Jacobian and a hyperbolic derivative which are not connected functions*, Rev. Math. Pures Appl. **9** (1964), 767–770.

9. I. Maximoff, *Sur les fonctions ayant la propriété de Darboux*, Prace Mat. Fiz. **43** (1936), 241–265.

10. L. Mišik, *Über die Funktionen der ersten Baireschen Klasse mit der Eigenschaft von Darboux*, Mat.-Fyz. Časopis Sloven. Akad. Vied **14** (1964), 44–49.

11. C. Neugebauer, *Darboux property for functions of several variables*, Trans. Amer. Math. Soc. **107** (1963), 30–37.

12. T. Tanaka, *On the family of connected subsets and the topology of spaces*, J. Math. Soc. Japan **7** (1955), 389–393.

13. J. Young, *A theorem in the theory of functions of a real variable*, Rend. Circ. Mat. Palermo **24** (1907), 187–192.

14. Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1–54.

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